3-Colorability Problem

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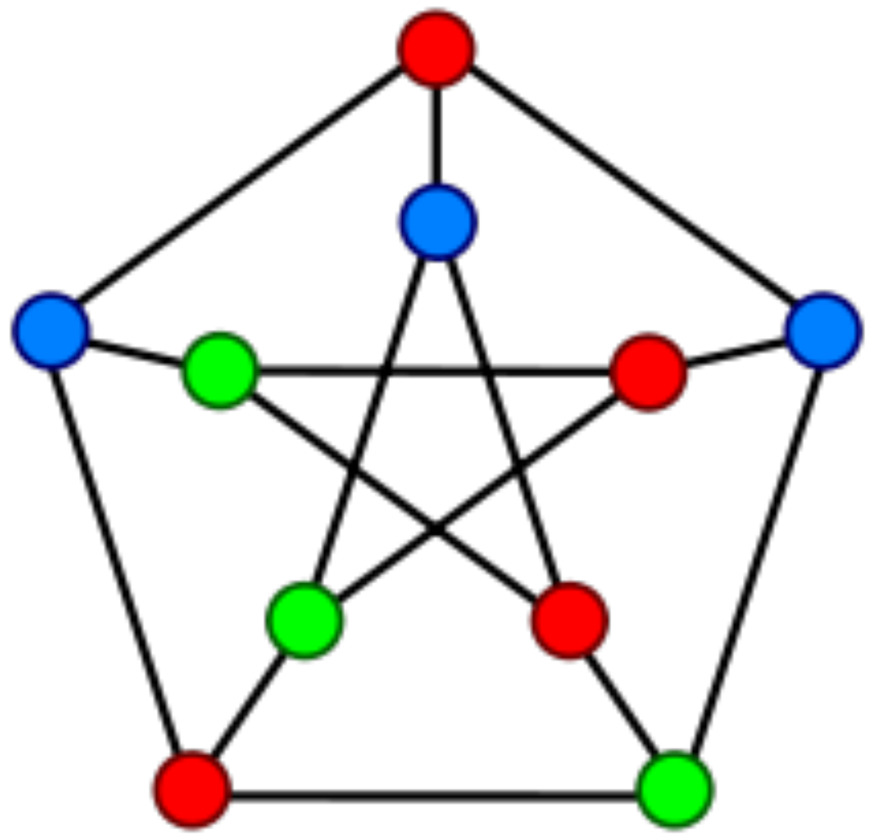
Problem Definition

Given a graph G with n vertices, color each vertex with given at most 3 colors in such a way that no adjacent vertices have the same color

Example



Given a graph G with 10 vertices and 3 colors (Blue, Green, Red), color each vertex without same vertices adjacent to one another

NP-complete

Condition

B is NP-complete if

1. B is in NP
2. Every A in NP is polynomial time reducible to B

3-Colorability is in NP

Proof

Given a non-deterministic polynomial time Turing machine for 3-colorability

1. Nondeterministically color a vertex with a nondeterministically selected color

2. Color the other vertices with different colors for the vertices adjacent to each other

3. If there are adjacent vertices having the same color, **reject**

4. If there are no adjacent vertices having the same color, **accept**

3-Colorability

Let B be the 3-colorability problem

B is NP-complete if

1. B is in NP
2. Every A in NP is polynomial time reducible to B

Polynomial Time Reducibility

$$A \leq p \; B$$

Where A is polynomial time reducible to B

To prove that 3-Colorability is NP-complete, 3-SAT can be used. Therefore,

$$\text{3-SAT} \leq p \; \text{3-Colorability}$$

3-SAT $\leq p$ 3-Colorability

Proof

Let ϕ be an instance of 3-SAT

$C_1, C_2, \ldots C_m$ be the clauses of ϕ

$X_1, X_2, \ldots X_n$ be the variables

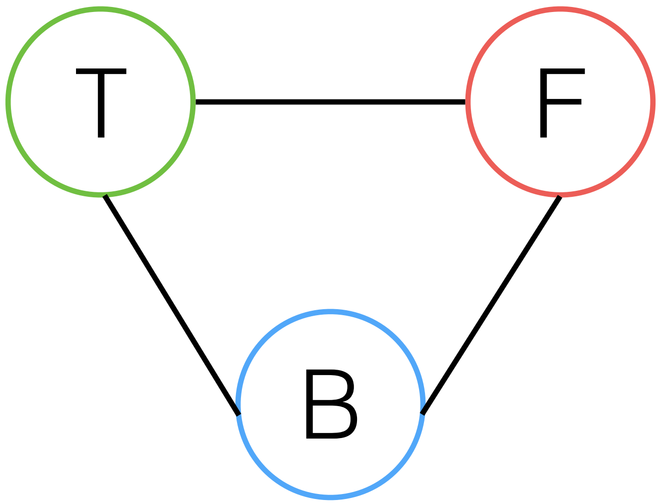
G(V, E) be the graph to be constructed from 3-SAT

3-SAT $\leq p$ 3-Colorability

Proof

1. Create a triangle in G with 3 vertices {T, F, B} where T stands for True, F stands for False and B stands for Base

2. Differently color each vertex. For example, color green, red and blue for the vertices T, F and B respectively

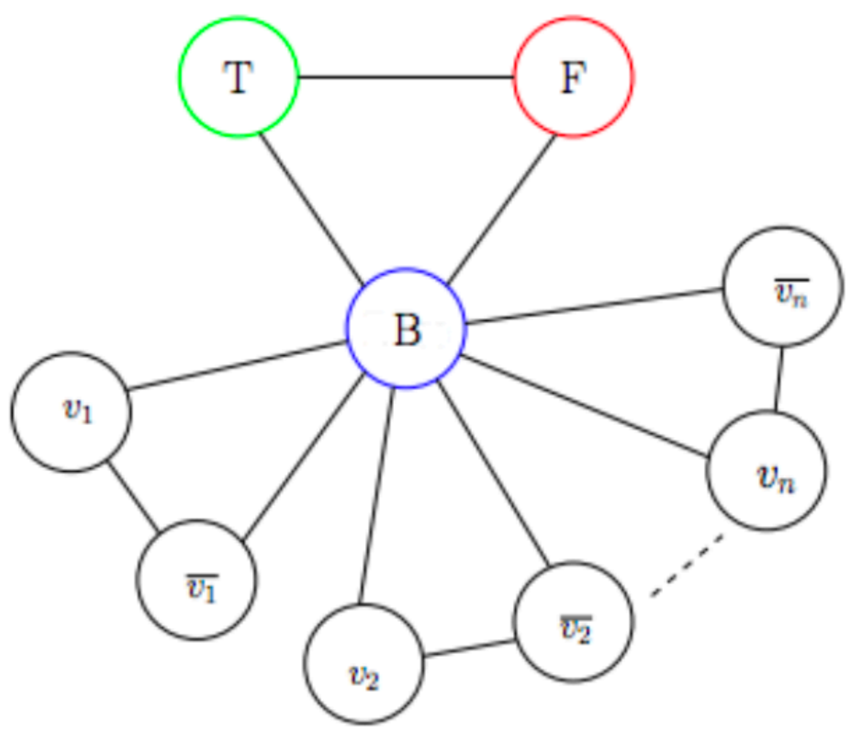


Proof of 3-Colorability as NP-complete

3-SAT $\leq p$ 3-Colorability

Proof

3. Add 2 vertices $v_i, \bar{v}_i$ for every literal X_i and create a triangle B, $v_i, \bar{v}_i$ for every pair of $v_i, \bar{v}_i$

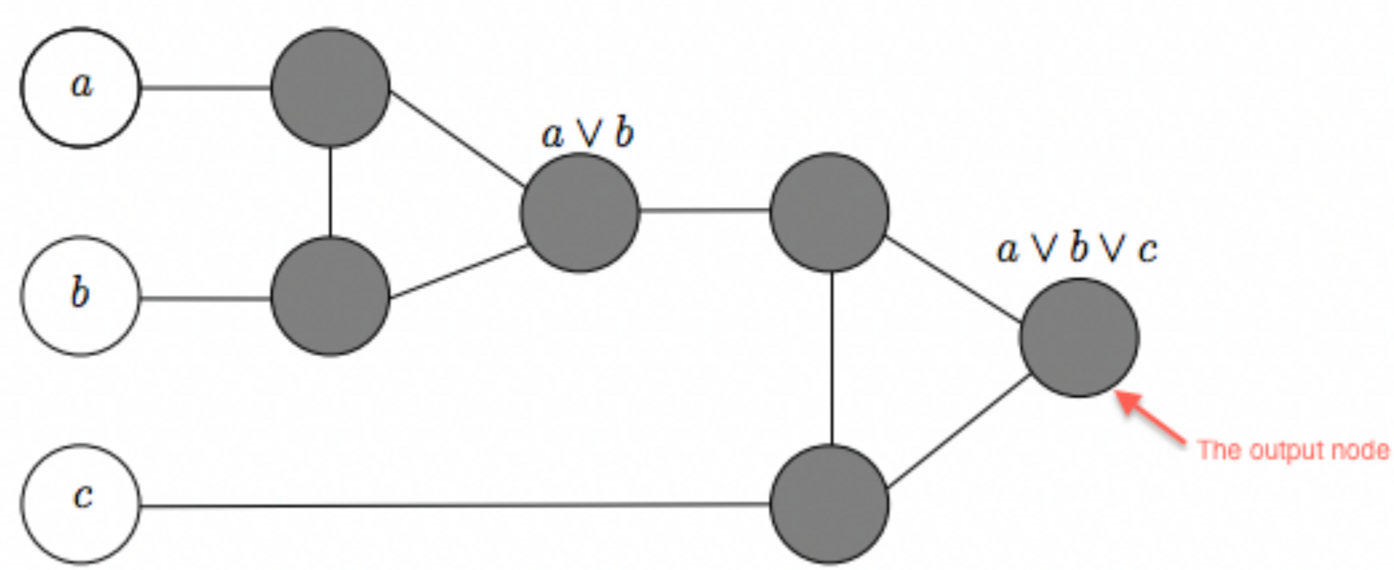


Proof of 3-Colorability as NP-complete

3-SAT $\leq p$ 3-Colorability

Proof

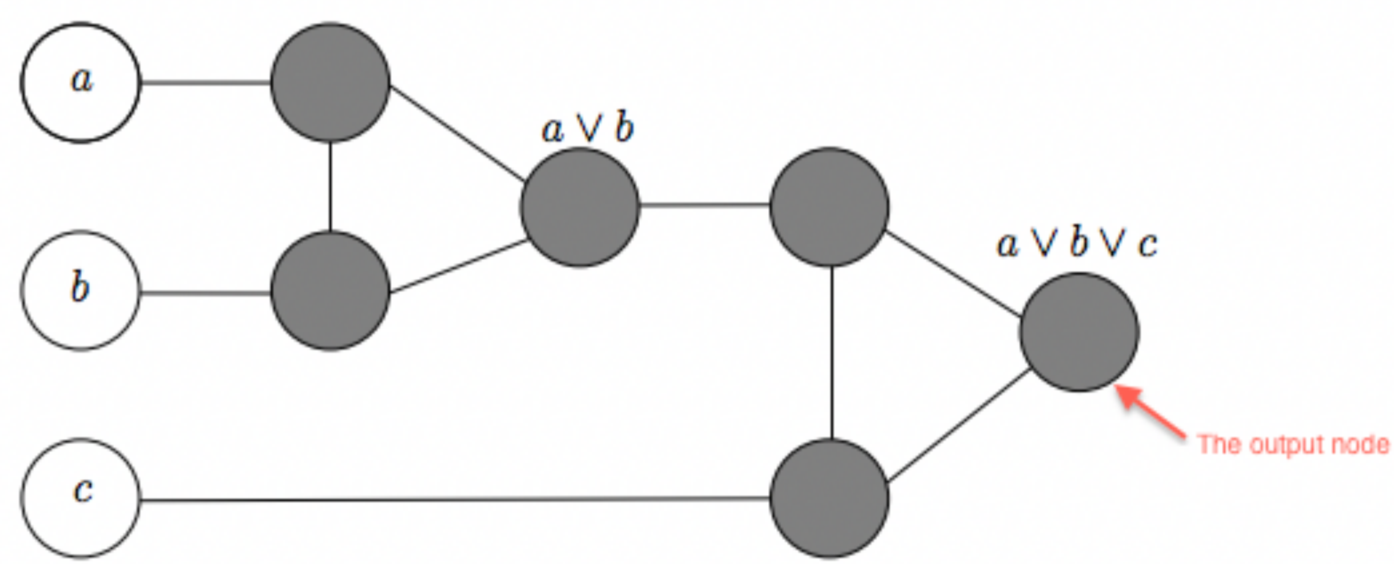
4. Construct the OR-gadget for each clause



3-SAT ≤ p 3-Colorability

Proof

5. Evaluate the output of the gadget with respect to the following conditions
    - If a, b, c are all false, color the output as F (red)
    - If one of a, b, c is true, color the output as T (green)

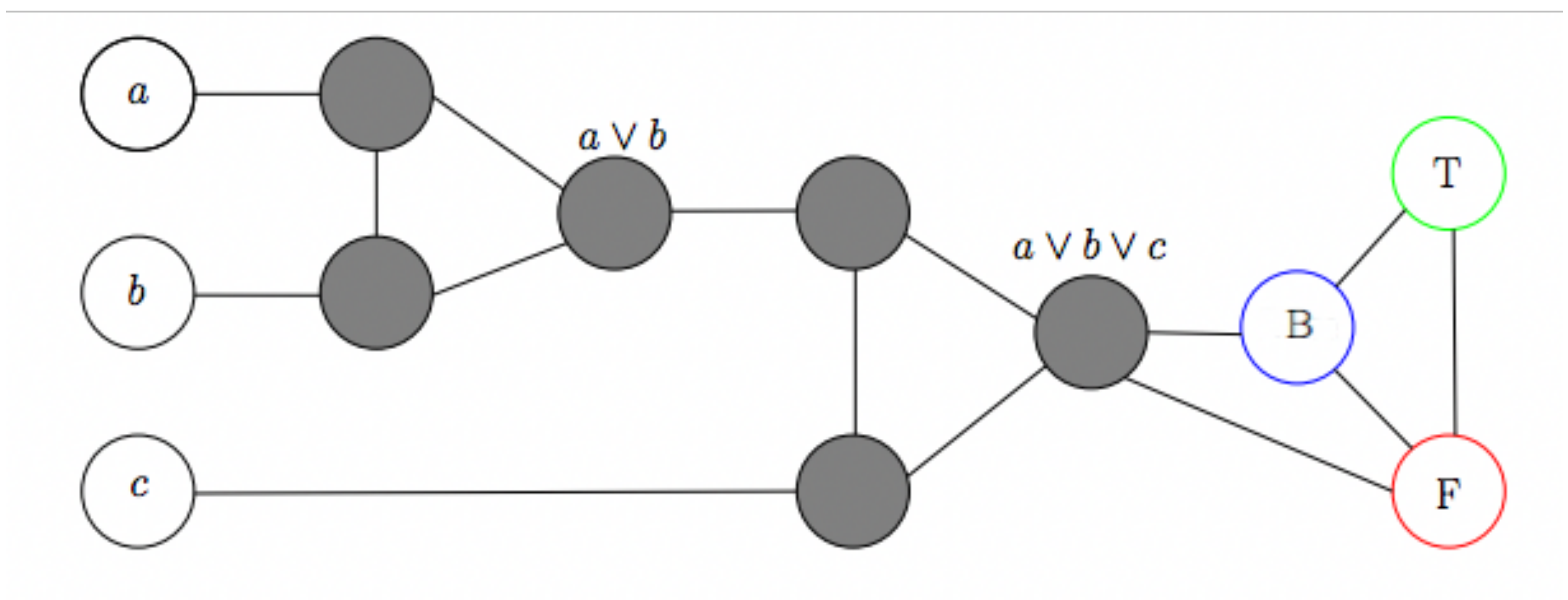


Proof of 3-Colorability as NP-complete

3-SAT ≤ p 3-Colorability

Proof

6. Connect the output to the B vertex and the F vertex of the initial triangle



Proof of 3-Colorability as NP-complete

3-Colorability

Let B be the 3-colorability problem
A be the 3-SAT problem

1. B is in NP ✓
2. Every A in NP is polynomial time reducible to B ✓

   3-SAT $\leq p$ 3-Colorability

3-Colorability is NP-complete

Reduction Time Complexity

1. Create a triangle in G with 3 vertices {T, F, B} where T stands for True, F stands for False and B stands for Base
—> **O(1)**

2. Differently color each vertex. For example, color green, red and blue for the vertices T, F and B respectively
—> **O(1)**

3. Add 2 vertices $v_i, \bar{v}_i$ for every literal X_i and create a triangle B, $v_i, \bar{v}_i$ for every pair of $v_i, \bar{v}_i$
—> **O(1)**

Reduction Time Complexity

For each clause *c* in All_Clauses

4. Construct the OR-gadget for the clause *c*

—> **O(1)**

5. Evaluate the output of the gadget with respect to the conditions

—> **O(1)**

6. Connect the output to the B vertex and the F vertex of the initial triangle

—> **O(1)**

7. If the connected nodes have the same color with their adjacent nodes, **reject**. Otherwise, **accept**.

—> **O(1)**

If there are n clauses, then the time complexity is **O(n)**

Example

Let us reduce the following 3-SAT to 3-Colorability

$$(a \cup b \cup c) \cap (\bar{a} \cup c \cup b) \cap (\bar{b} \cup \bar{c} \cup a)$$

Assuming that, a is true, b is true and c is false

$(a \cup b \cup c)$ → True

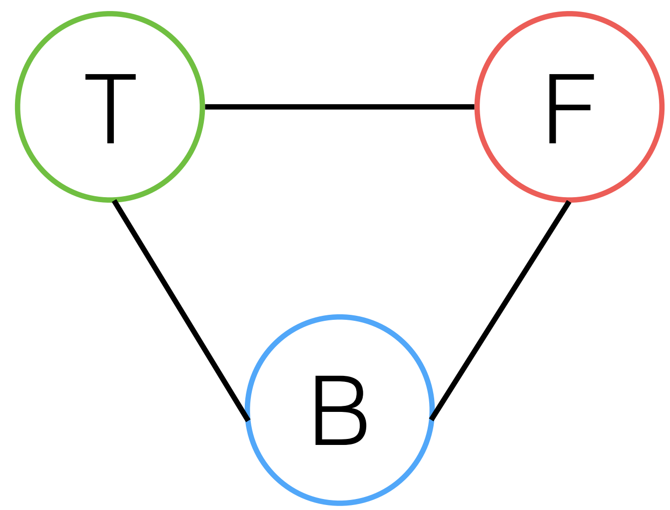
$(\bar{a} \cup c \cup b)$ → True

$(\bar{a} \cup c \cup b)$ → True

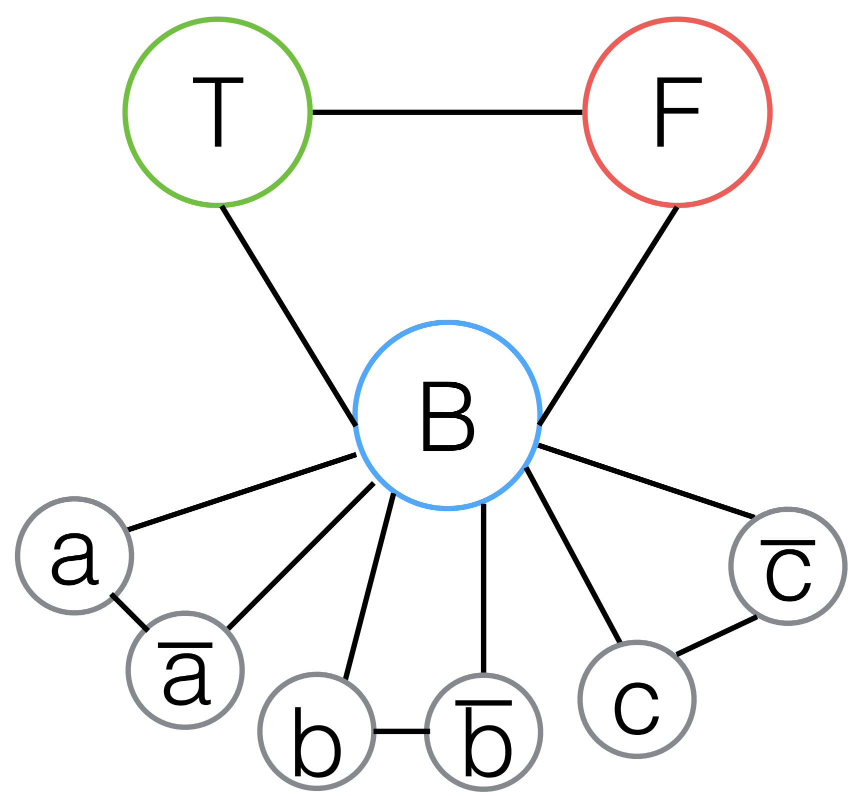
SATISFIED ✓

3-SAT ≤ p 3-Colorability

Step 1



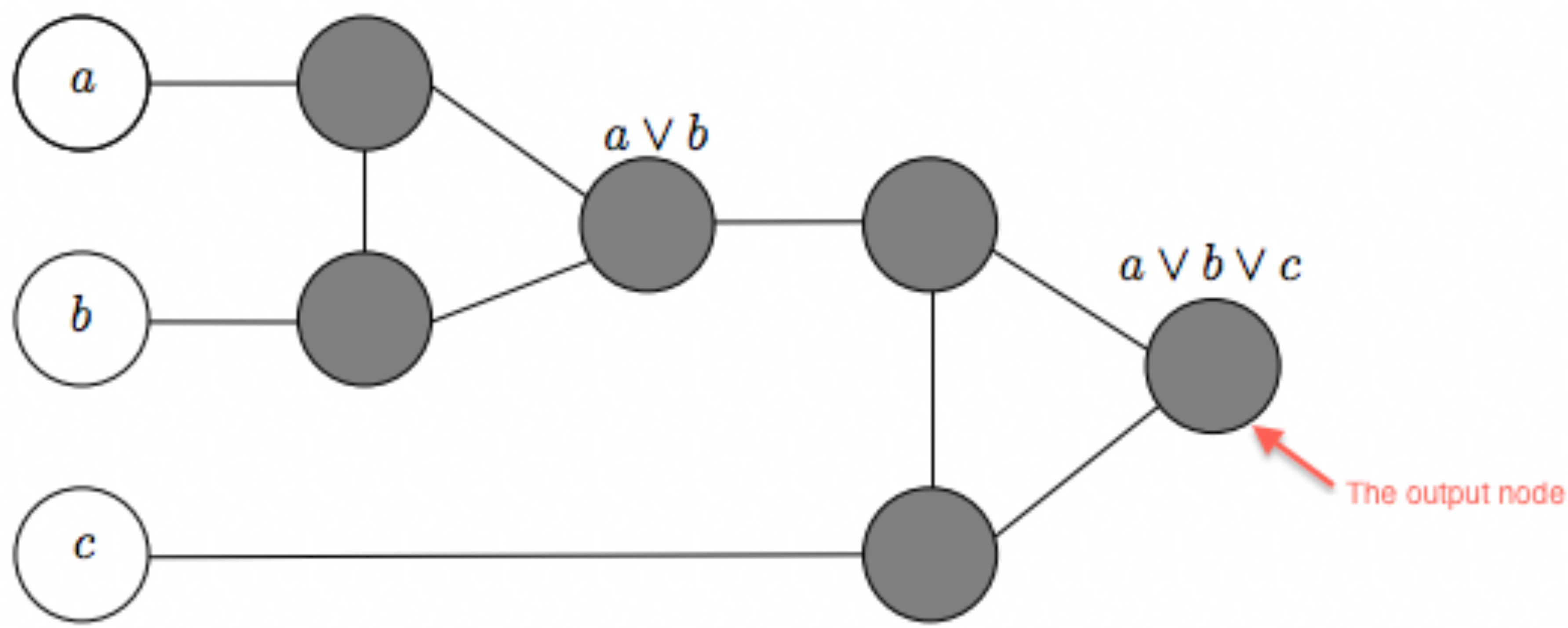
Step 2



Proof of 3-Colorability as NP-complete

3-SAT $\leq p$ 3-Colorability

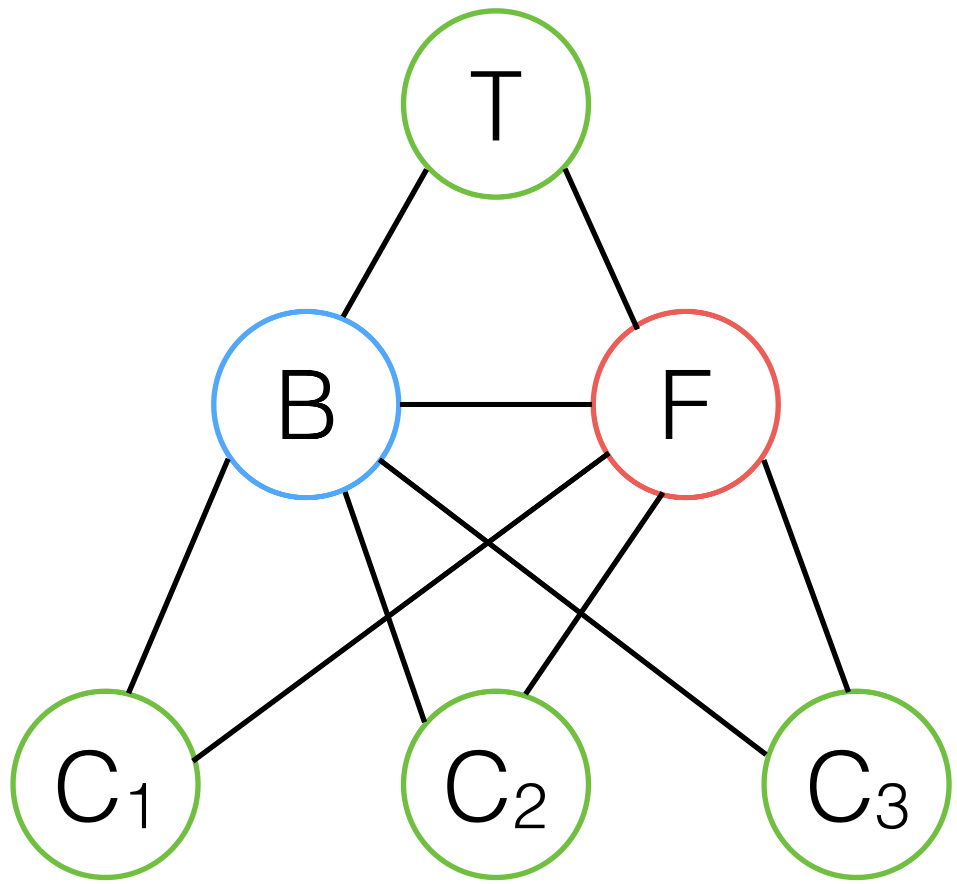
Step 3



Proof of 3-Colorability as NP-complete

3-SAT ≤ p 3-Colorability

Step 4



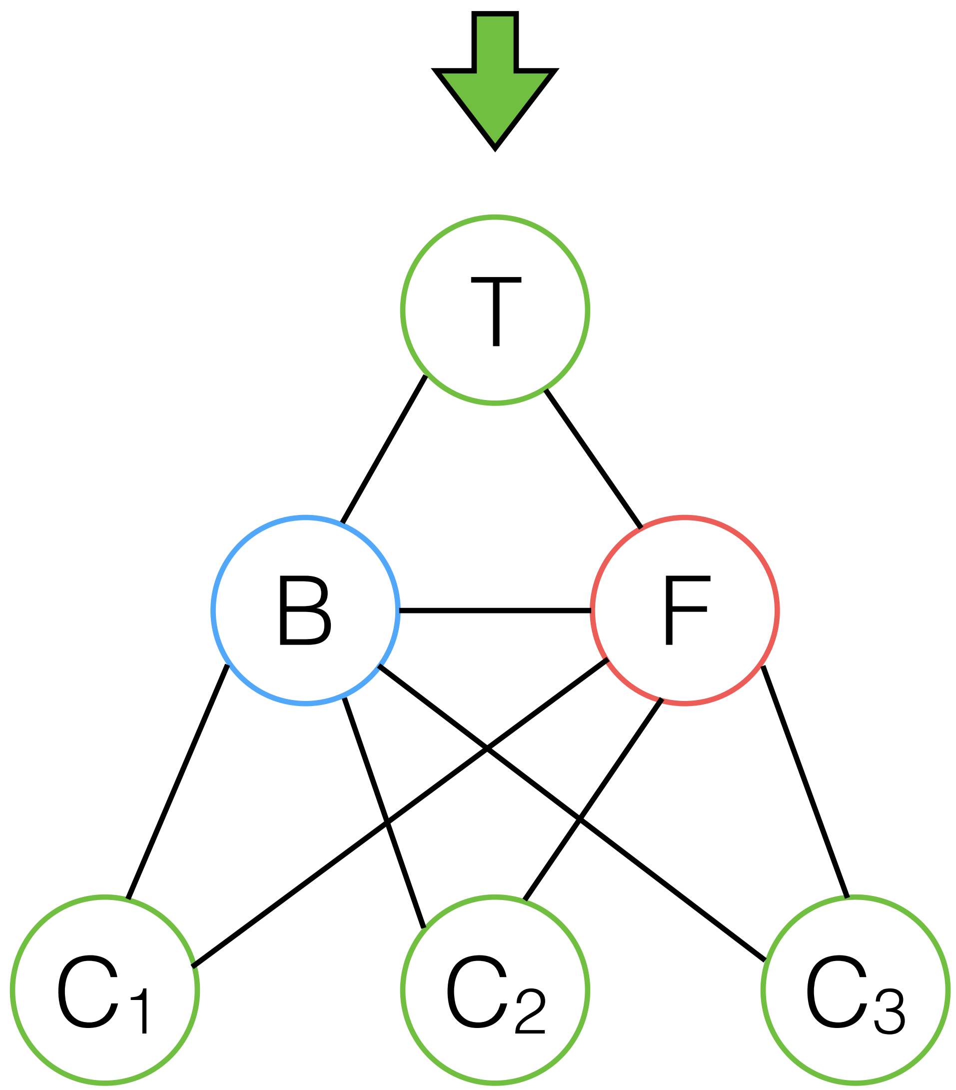
$(a \cup b \cup c)$

$(\bar{a} \cup c \cup b)$

$(\bar{a} \cup c \cup b)$

3-SAT $\leq p$ 3-Colorability

$$(a \cup b \cup c) \cap (\bar{a} \cup c \cup b) \cap (\bar{b} \cup \bar{c} \cup a)$$



Proof of 3-Colorability as NP-complete

Thank You